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An Inhomogeneous Couette-Type Flow with a Perfect Slip Condition at the Lower Boundary of an Infinite Fluid Layer

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Abstract. In this paper, we investigate an exact solution to the three-dimensional problem of an isobaric flow of a viscous incompressible fluid layer. The solution is a linear function of the longitudinal coordinates. The solution type under study describes a vertical twist in the fluid, which arises due to the inclusion of inertial forces and inhomogeneous velocity distribution at the free boundary of the fluid layer. The solution allows to us describe the counterflow of an incompressible fluid in a thin layer. The exact solution is obtained for the perfect slip condition at one of the boundaries of the fluid layer. Conditions for the existence of points inside the fluid layer at which the velocity vanishes are defined.

PROBLEM STATEMENT

To describe the process under study, we write the Navier-Stokes equation for the steady-state motion of a viscous fluid and the incompressibility equation. The equations are projected onto the axes of a Cartesian coordinate system. The stationary system of nonlinear partial differential equations has the following form [1]:

$$\begin{gathered}
V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} = -\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right),\\
V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_y}{\partial z} = -\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right),\\
V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z} = -\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right),\\
\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0 .
\end{gathered} \tag{1}$$

Here, V_x, V_y, and V_z are the components of the velocity vector; P is the deviation of pressure from hydrostatic, taken relative to the constant average density of the fluid ρ; ν is kinematic (molecular or turbulent) viscosity [2]. The exact solution of system (1) is sought in the following form [3–6]:

$$\begin{gathered}
V_x = U(z) + yu(z),\ V_y = V(z),\ V_z = w(z),\\
P = P_0(z) .
\end{gathered} \tag{2}$$

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We substitute the class of exact solutions (2) into the nonlinear system (1). The equation system is written in the following form:

$$
\begin{gathered}
Vu + w\left(\frac{\partial U}{\partial z} + \frac{\partial u}{\partial z} y\right) = \nu\left(\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 u}{\partial z^2} y\right), \\
w\frac{\partial V}{\partial z} = \nu\frac{\partial^2 V}{\partial z^2}, \\
w\frac{\partial w}{\partial z} = -\frac{\partial P_0}{\partial z} + \nu\frac{\partial^2 w}{\partial z^2}, \\
\frac{\partial w}{\partial z} = 0 .
\end{gathered}
\tag{3}
$$

It follows from expressions (3) obtained for the type of solutions (2) that the vertical component of velocity V_z and pressure P are constant functions defined, for example, at the boundary of a fluid layer. Pressure can be taken equal to atmospheric and specified on the upper free surface. Thus, in system (3), the last two equations determined by the boundary conditions are automatically satisfied. In what follows, we solve only the first two equations of system (3).

We mark the partial derivatives in system (3) by a prime since all the required functions depend only on z and reduce the system to a dimensionless form. We choose the following scale variables: the horizontal coordinates x and y characterized by the scale l and the vertical coordinate z characterized by the fluid layer thickness h. The scale of the horizontal velocities V_x and V_y is denoted by $[U]$.

The dimensionless variables x, y, and z are further denoted by the same letters. After the reduction of system (3) to a dimensionless form, the equations become as follows:

$$
\begin{gathered}
U'' - \mathrm{Re}_w U' - \delta^2 \mathrm{Re}_U Vu = 0, \\
u'' - \mathrm{Re}_w u' = 0, \\
V'' - \mathrm{Re}_w V' = 0 .
\end{gathered}
\tag{4}
$$

Here, $\mathrm{Re}_U = \frac{[U]l}{\nu}$ and $\mathrm{Re}_w = \frac{wh}{\nu}$ are the dimensionless Reynolds numbers, with the subscripts indicating for which velocity the dimensionless complex is introduced; $\delta = h/l$ is the anisotropy parameter, the ratio of vertical to horizontal characteristic dimensions.

THE BOUNDARY VALUE PROBLEM WITH A PERFECT SLIP CONDITION AT THE LOWER BOUNDARY

We consider a boundary value problem for equation system (4) with the perfect slip condition [1, 7] at the lower solid boundary of the fluid layer. The boundary value problem describes the flow of a viscous incompressible fluid in an infinitely long horizontal layer of thickness h. Parabolic wind is specified at the upper free boundary as [3, 8]

$$
\begin{gathered}
\left.\frac{\partial U}{\partial z}\right|_{z=0} = 0, \quad \left.\frac{\partial u}{\partial z}\right|_{z=0} = 0, \quad \left.\frac{\partial V}{\partial z}\right|_{z=0} = 0, \\
U(1) = \cos\varphi, \quad u(1) = \frac{\mathrm{Ta}}{2\,\mathrm{Re}_U}, \quad V(1) = \sin\varphi .
\end{gathered}
\tag{5}
$$

Here, φ is an arbitrary angle; $\mathrm{Ta} = 2\Omega l^2/\nu$ is the modified Taylor number [9]; Ω is the vertical component of vorticity.

The exact particular solution of the boundary value problem (4), (5) has the form

$$u = \frac{\mathrm{Ta}}{2\mathrm{Re}_U}, \qquad V = \sin\varphi,$$

$$U = \cos\varphi - \frac{\delta^2 \mathrm{Ta}\sin\varphi}{2\mathrm{Re}_w^2}\left[\exp\left(\mathrm{Re}_w\right) - \exp\left(\mathrm{Re}_w z\right) + \mathrm{Re}_w\left(z-1\right)\right]. \tag{6}$$

The function U may have at most one root on the interval $z \in (0;1)$ if boundary conditions (5) are satisfied; taking into account the fact that the other velocity components u and V are constant functions in this case, we can make the following conclusion. If the perfect slip condition at the lower boundary and conditions (5) at the upper boundary are fulfilled, the layer of a viscous incompressible fluid under study can have at most one stagnant point.

The streamlines for the case of a perfect slip condition at the lower boundary are shown in Fig. 1. The changing direction of the fluid flow shows the existence of stagnant points.

The localization of the roots of the function U in the interval $z \in (0;1)$ for various values of the modified Taylor number Ta is shown in Fig. 2. From the physical meaning of the Taylor number, we find that the zero value of the function U will localize near the upper boundary as the centrifugal forces increase significantly compared to the viscous friction forces

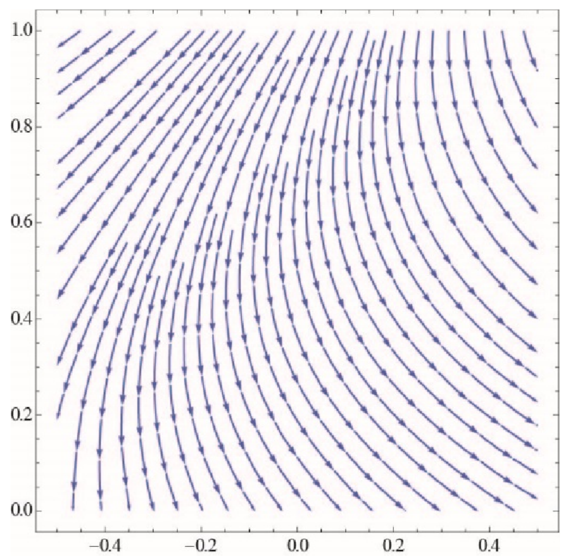

FIGURE 1. The stream lines for $\delta = 0.01$, $\varphi = -2\pi/3$, $\mathrm{Re}_U = 1\cdot 10^5$, $\mathrm{Re}_w = -10$, $\mathrm{Ta} = 3\cdot 10^5$

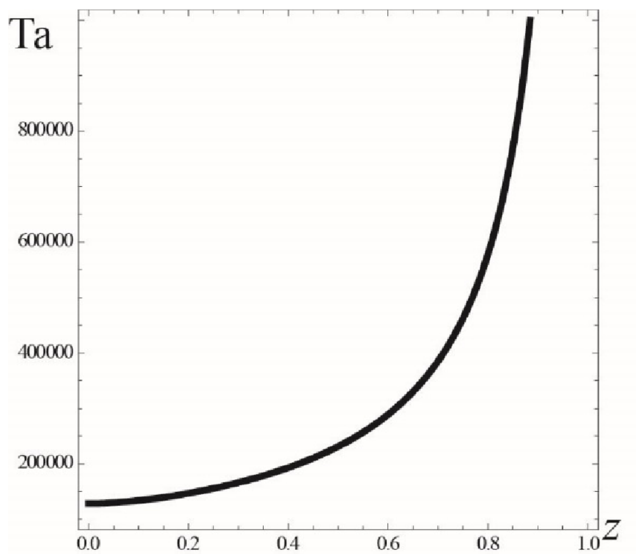

FIGURE 2. The localization of the roots of the function U for the values of the modified Taylor number $\mathrm{Ta} \in \left[10^5;10^6\right]$ when $\delta = 0.01$, $\varphi = -2\pi/3$, $\mathrm{Re}_w = -10$

Additional research shows the existence of counterflow points of the velocity component U for various values of the angle φ and the similarity parameters Re_w and Ta. For example, for the angle $\varphi = -2\pi/3$, we obtain the root existence region shown in Fig. 3. The following values of the similarity parameters are considered: $\mathrm{Re}_w \in [-11;5]$, $\mathrm{Ta} \in [1\,000;150\,000]$. Figure 4 shows the regions of the existence of counterflow points for the velocity component U at the fixed Reynolds number $\mathrm{Re}_w = -1$, with different angles $\varphi \in [0;2\pi]$ and different modified Taylor numbers $\mathrm{Ta} \in [1\,000;200\,000]$.

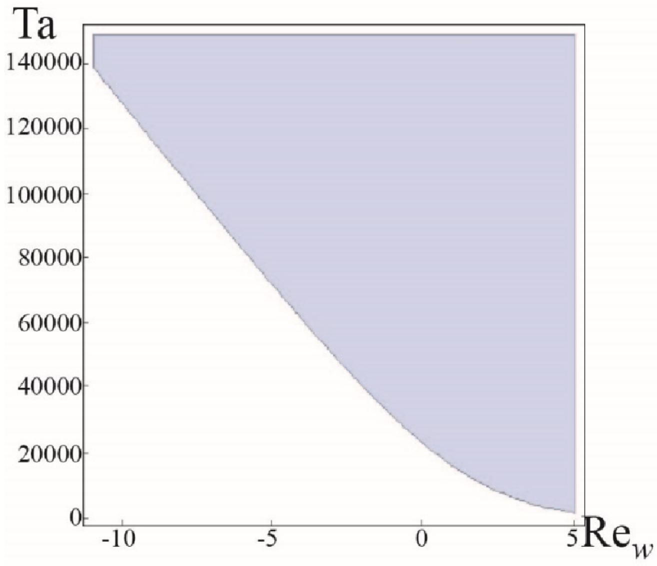

FIGURE 3. The region of the existence of counterflow points for the velocity component U at fixed values of the angle $\varphi = -\frac{2\pi}{3}$ and different values of $Re_w \in [-11;5]$ and $Ta \in [10^3;15 \cdot 10^4]$

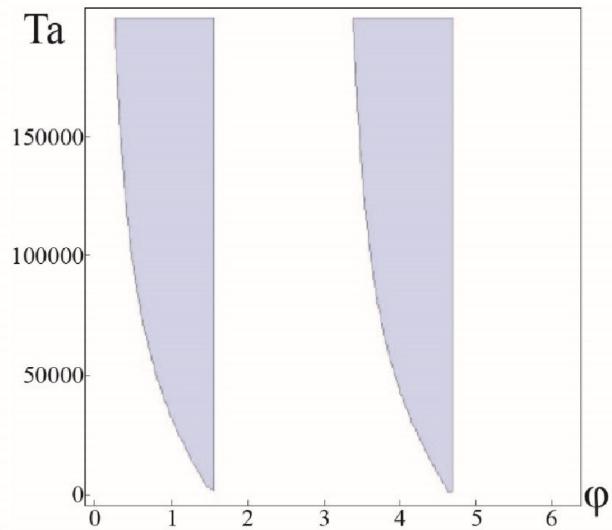

FIGURE 4. The regions of the existence of counterflow points for the velocity component U at fixed values of $Re_w = -1$ and different values of $\varphi \in [0;2\pi]$ and $Ta \in [10^3;2 \cdot 10^5]$

CONCLUSION

In this paper, a generalization of the steady-state classical Couette flow has been obtained for three-dimensional nonlinear viscous incompressible fluids.

It has been demonstrated that, in an isobaric flow, counterflow can occur in a fluid layer when exact solutions from the class of linearly increasing velocities in horizontal coordinates are considered with account taken of the perfect slip condition. The localization of stratification in the near-boundary layers of the fluid flow is shown. The analysis of the solutions is applicable to large-scale currents in the World Ocean.

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